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ADAPTIVE MINIMUM MEAN SQUARE ERROR FORECAST OF MISSILE TRAJECTO--ETC(U)
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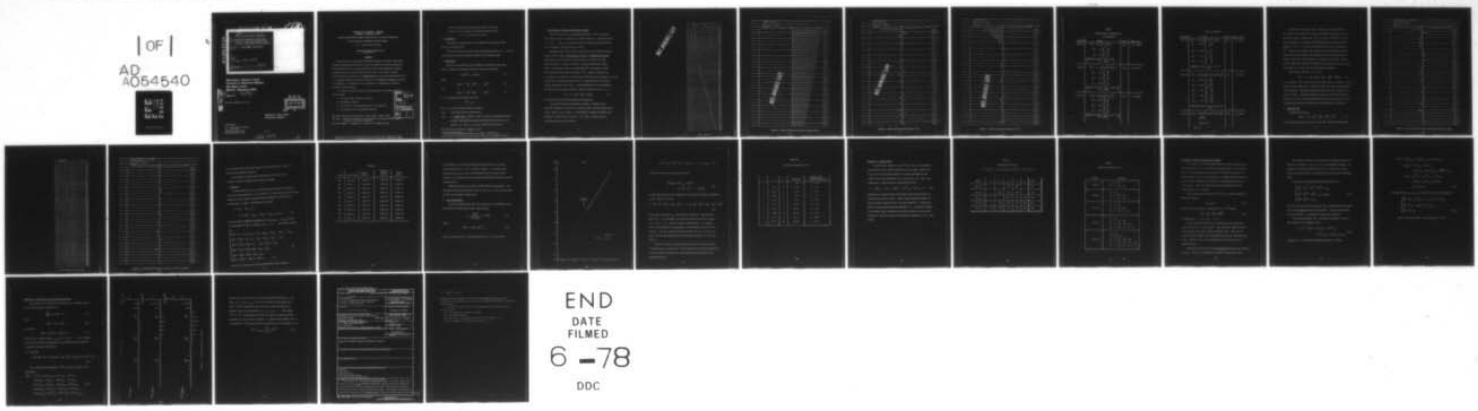

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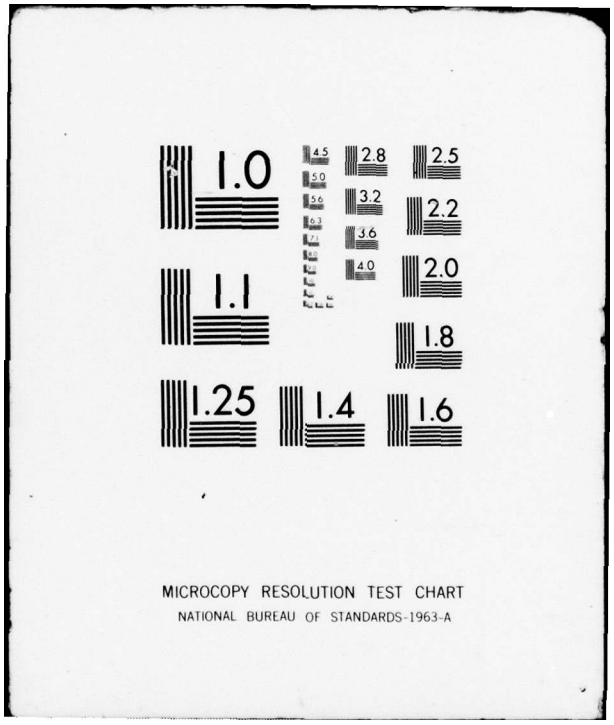
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(6) ADAPTIVE MINIMUM MEAN SQUARE ERROR FORECAST OF MISSILE TRAJECTORY USING STOCHASTIC DIFFERENCE EQUATION MODELS.

(10) G. E. P. Box and Lars Pallesen

(14) MRC-TSR-1821

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G. E. P. Box and Lars Pallesen

Technical Summary Report #1821
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ABSTRACT

Data projection (forecasting) methods developed in the book "Time Series Analysis Forecasting and Control" by Box and Jenkins (published by Holden-Day revised edition 1976) are illustrated using missile data supplied by Quality Evaluation Division of White Sands Missile Range. The process of model building making iterative use of identification (using the autocorrelation function) fitting (maximum likelihood estimation) and diagnostic checking (analysis of residuals) is illustrated in the building of an appropriate stochastic difference equation model.

It is shown in detail how all the following may be calculated directly from the model

1) the projection (forecast) function
2) the memory function
3) the error of the projected value at any lead time
4) the updating of the projection function.

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AMS (MOS) Subject Classifications: 60G25, 62M10, 62M20, 62N15

Key Words: Missile, Trajectory, Data projection, Forecasting ARCMA models, Stochastic difference equations

Work Unit Number 4 (Probability, Statistics and Combinatorics)

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ADAPTIVE MINIMUM MEAN SQUARE ERROR FORECAST OF MISSILE TRAJECTORY USING STOCHASTIC DIFFERENCE EQUATION MODELS

G. E. P. Box and Lars Pallesen

1. Introduction

The objective of this report is to illustrate data projection using the Box and Jenkins text*.

The data used for illustration here were furnished by Mr. Paul H. Thrasher of the Quality Evaluation Division of White Sands Missile range.

2. Model Form

Reasons are presented in B&J Chapter 4 for employing time series models, which are stochastic difference equations of the form

$$\phi_p(B)\,\nabla^d z_t = \theta_q(B)\,a_t \tag{1}$$

where

$$\phi_p(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p \tag{2}$$

and

$$\phi_q(B) = 1 - \theta_1 B - \theta_2 B^2 - \ldots - \theta_q B^q \tag{3}$$

B is a backward shift operater such that

$$Bz_t = z_{t-1} .$$

$\nabla = 1 - B$ is the backward difference operator.

$\{z_t\}$ is the time series of observations.

$\{a_t\}$ is a white noise sequence, that is a series of independent random variables approximately normally distributed with mean zero and variance σ_a^2. The a_t's are also called random shocks.

The model (1) is said to be of order (p, d, q)

* Time Series Analysis Forecasting and Control", Holden-Day.

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

3. Identification, Fitting and Checking of Model

The data series we are considering consists of 246 consecutive observations of the x-coordinate of a missile trajectory. The observations, z_t; $t = 1, 2, \ldots, 246$, were made with constant sampling interval and there are no missing or obviously aberrant values.

Modeling such a time series is conceived of as an iterative process involving three stages: identification, fitting and diagnostic checking. Identification is first performed along the lines of Chapter 6 in B&J. Plotting the data z_t (Figure 1a) shows a smooth nonstationary series, whose autocorrelation function (Figure 1(b)) dies out extremely slowly. After differencing three times the series $\nabla^3 z_t$ appears stationary and its sample autocorrelation and partial autocorrelation function (Figures 1c and 1d) suggest that a reasonable model for $\nabla^3 z_t$ should include a few moving average parameters of low order. A clear identification is not possible at this point but a stochastic difference equation model of order $(0, 3, 3)$

$$\nabla^3 z_t = (1 - \theta_1 B - \theta_2 B^2 - \theta_3 B^3) a_t \qquad (4)$$

is considered worthy of being tentatively entertained.

Fitting this model by the method of Chapter 7 in B&J gives the parameter estimates, residual sum of squares (RSS) and the residual mean square (RMS) listed in Table I. If this model is adequate the RMS value provides an estimate of the variance, $\sigma_a^2 = E(a_t^2)$, which is the one step ahead forecast error variance.

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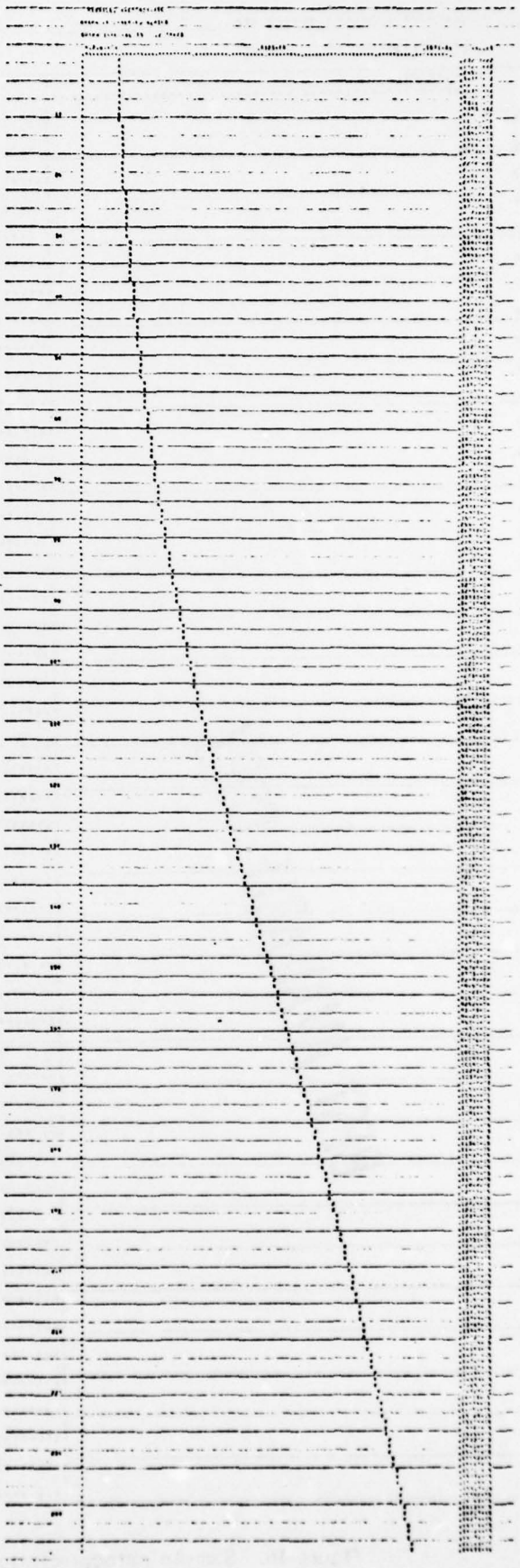

Figure 1a. Plot of data

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Figure 1b. Sample autocorrelation function of original series

MISSILE, X-COORDINATE

GRAPH OF DIFFERENCE 3 ACF

GRAPH INTERVAL IS .2000-01

	-.1000+01 .0000 .1000+01	VALUES
1	XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX	-.86337+00
2	XXXXX	.77686-01
3	XXXXXXXXXX	.18781+00
4	XXXXXXXX	-.14199+00
5	XXX	.69709-01
6	XX	-.21137-01
7	XXX	.39764-01
8	XXXX	-.80137-01
9	XXXXX	.72670-01
10	XXX	-.40234-01
11	XX	-.11794-01
12	XXX	.35[illegible]
13	XXX	-.30375-01
14	XX	.23377-01
15	X	.29[illegible]
16	XXXX	-.50551-01
17	XXXXXX	.91971-01
18	XXXXX	-.83[illegible]
19	XX	.20423-01
20	XXX	.46512-01
21	XXX	-.37137-01
22	XXX	-.45515-01
23	XXXXXXX	.13[illegible]
24	XXXXXXX	-.11[illegible]
25	XXX	.47[illegible]
26	XX	.10620-01
27	XXXX	-.55634-01
28	XXXX	.64[illegible]
29	XXX	-.32235-01
30	XX	-.23439-01
31	XXX	.33[illegible]
32	X	-.14[illegible]
33	XX	.11[illegible]
34	XXXX	-.[illegible]
35	XXXXXX	.96[illegible]
36	XXXX	-.54[illegible]
37	XXX	-.37[illegible]
38	XXXXXX	.10553+00
39	XXXXX	-.76[illegible]
40	XX	-.23[illegible]
41	XXXX	.51237-01
42	XXX	.42[illegible]
43	XXXXXXXX	-.14035+00
44	XXXXXXXX	.13[illegible]
45	XXXXX	-.[illegible]
46	XXXX	.50432-01
47	XXX	-.39[illegible]
48	XX	.28[illegible]
49	XXXX	-.52602-01
50	XXXXX	.89155-01

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Figure 1c. Sample autocorrelation function for $\nabla^3 z_t$

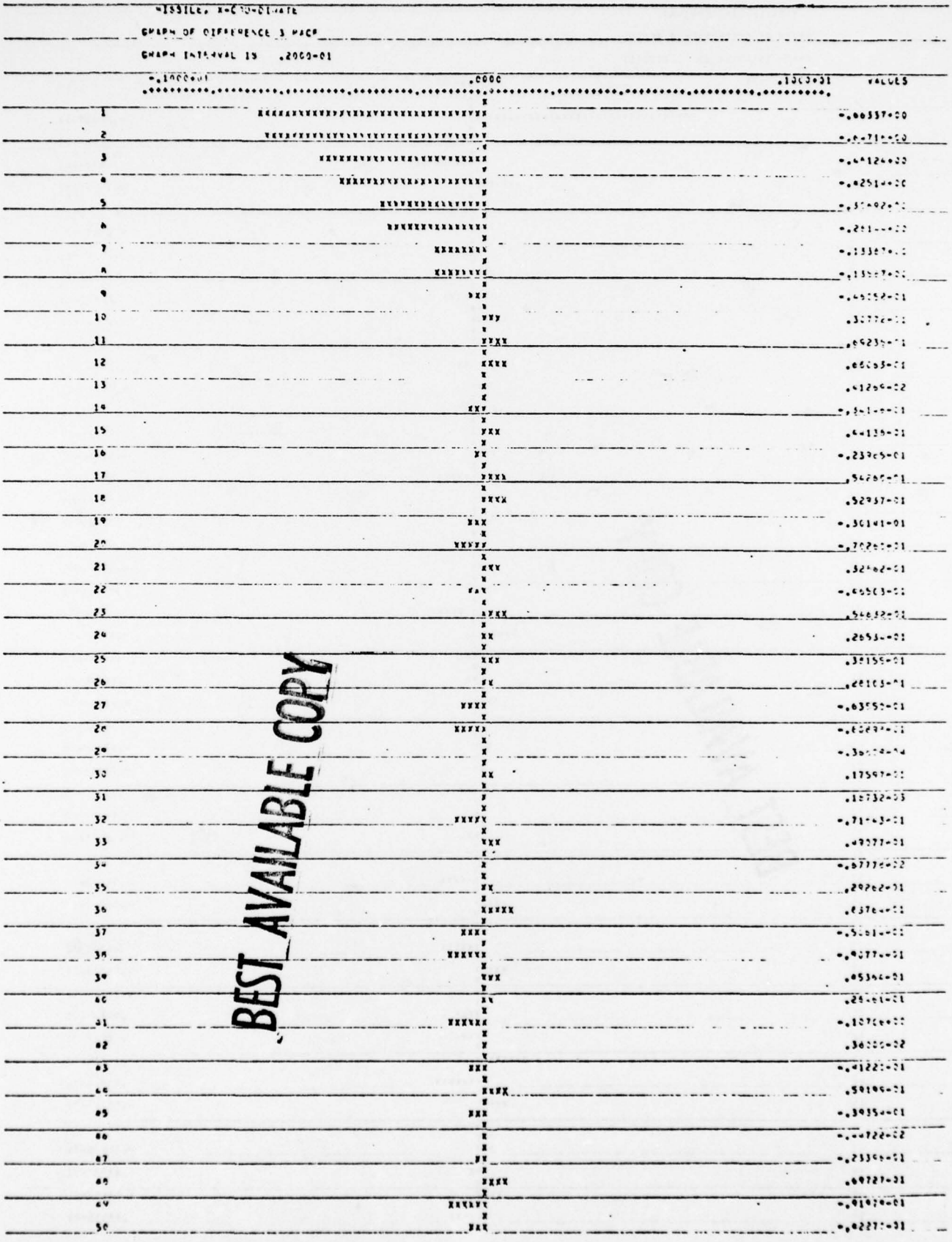

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Figure 1d. Partial autocorrelation function for $\nabla^3 z_t$

Table I

Models fitted to Missile data

(x-coordinate)

(p,d,q)	Model	RSS	RMS (DF)
(0, 2, 2)	$\nabla^2 z_t = (1 - \theta_1 B - \theta_2 B^2)a_t$ $\hat{\theta}_1 = .716 \begin{cases} .83 \\ .60 \end{cases}$ $\hat{\theta}_2 = -.517 \begin{cases} -.40 \\ -.63 \end{cases}$ (Moduli of roots: 1.39; 1.39 i.e. stable)	410.	1.69 (242)
(0, 2, 3)	$\nabla^2 z_t = (1 - \theta_1 B - \theta_2 B^2 - \theta_3 B^3)a_t$ $\hat{\theta}_1 = .662 \begin{cases} .78 \\ .54 \end{cases}$ $\hat{\theta}_2 = -.114 \begin{cases} .03 \\ -.26 \end{cases}$ $\hat{\theta}_3 = -.425 \begin{cases} -.31 \\ -.54 \end{cases}$ (Moduli of roots: 1.14; 1.14; 1.83 i.e. stable)	348.	1.44 (241)
(0, 3, 3)	$\nabla^3 z_t = (1 - \theta_1 B - \theta_2 B^2 - \theta_3 B^3)a_t$ $\hat{\theta}_1 = 1.731 \begin{cases} 1.75 \\ 1.72 \end{cases}$ $\hat{\theta}_2 = -.776 \begin{cases} -.76 \\ -.79 \end{cases}$ $\hat{\theta}_3 = -.104 \begin{cases} -.08 \\ -.13 \end{cases}$ (Moduli of roots: 1.014; 1.014; 9.39 i.e. stable)	247.	1.03 (240)

Table I Continued

(p,d,q)	Model	RSS	RMS (DF)
(0, 3, 4)	$\nabla^3 z_t = (1-\theta_1 B-\theta_2 B^2-\theta_3 B^3-\theta_4 B^4)a_t$ $\hat{\theta}_1 = 1.938$ { 1.99, 1.89 } $\hat{\theta}_2 = -1.030$ { -.95, -1.11 } $\hat{\theta}_3 = -.146$ { -.02, -.27 } $\hat{\theta}_4 = .173$ { .25, .09 } (Moduli of roots: 1.13; 1.13; 1.59; 2.86)	203.	.85 (239)
(0, 3, 5)	$\nabla^3 z_t = (1-\theta_1 B-\theta_2 B^2-\theta_3 B^3-\theta_4 B^4-\theta_5 B^5)$ $\hat{\theta}_1 = 2.078$ { 2.11, 2.04 } $\hat{\theta}_2 = -1.291$ { -1.21, -1.37 } $\hat{\theta}_3 = -.115$ { .00, -.23 } $\hat{\theta}_4 = .395$ { .51, .28 } $\hat{\theta}_5 = -.131$ { -.04, -.22 } (Moduli of roots: 1.11; 1.11; 1.79; 1.79 1.93)	192.	.81 (238)
(0, 3, 6)	$\nabla^3 z_t = (1-\theta_1 B-\theta^2 B_2-\theta_3 B^3-\theta_4 B^4-\theta_5 B^5$ $-\theta_6 B^6)a_t$ (roots o. k.) $\hat{\theta}_6 \approx 0$	192.	.81 (237)

Diagnostic checking (Chapter 8 in B&J) involves examination of the residuals (the estimated a_t's) left after fitting this model to seek for departures from the "white noise" form. One way of doing this is to submit the residual $\hat{a}_t$ sequence to the identification procedure previously applied to $\nabla^3 z_t$. In fact the autocorrelation function of the residuals $\hat{a}_t$'s, Figure 2(a), suggests that while most of the dependence is being accounted for by the model, some significant low order autocorrelation remains, indicating some additional θ parameters are needed. Notice, that the diagnostic checking of the model (4) reveals model inadequacy and also identifies in which way the model should be modified.

After another cycle the (0,3,5) model

$$\nabla^3 z_t = (1 - \theta_1 B - \theta_2 B^2 - \theta_3 B^3 - \theta_4 B^4 - \theta_5 B^5)a_t \qquad (5)$$

is considered, and it fits the data very well, leaving residuals, Figure 2(b), which look like white noise. Figure 2(c) shows the sample autocorrelations of the residuals. This fitted model along with some other contenders are listed in Table I. Additional models are fitted as a check that additional parameters would not substantially improve matters (overfitting), and also to demonstrate that the chosen number of differencings is appropriate.

4. Checking $\theta(B)$

Regarding the operator

$$\theta(B) = 1 - \theta B - \theta_2 B^2 - \theta_3 B^3 - \theta_4 B^4 - \theta_5 B^5 \qquad (6)$$

as a polynomial in B, it is shown in B&J that a necessary requirement

THE ESTIMATED RESIDUALS - MODEL (0,3,3)

GRAPH OF OBSERVED SERIES ACF

GRAPH INTERVAL IS .2000-01

	-.1000+01 .0000 .1000+01	VALUES
1	XXXXXXXXXXXXX	-.25040+00
2	XXX	-.68849-01
3	XXXXXXXXXXX	.19643+00
4	XXXX	-.62422-01
5	XXX	-.35029-01
6	XXXXX	-.71012-01
7	XXXXX	-.72554-01
8	XXXXXXXX	-.13262-00
9	XXXX	-.51819-01
10	XXXXXX	-.99534-01
11	XXXX	-.57213-01
12	XX	.26498-01
13	XX	.29367-01
14	XXXX	.69692-01
15	XXXX	.52921-01
16	XXX	.41542-01
17	XXXXXXXX	.13572-00
18		.66678-02
19	XXXX	.65529-01
20	XXXXX	.71035-01
21	XXXX	-.59214-01
22	XXXX	-.61548-01
23	XXX	.38356-01
24	XXXXXXX	-.12411-00
25	XX	-.24667-01
26	XX	-.16122-01
27	XXXX	-.50179-01
28	XX	.25043-01
29	XXX	-.86543-01
30	XX	-.29327-01
31	XXXX	.62960-01
32	XXXX	.50620-01
33	XX	.10159-01
34	XXX	-.30966-01
35	XXXXX	.82122-01
36	XX	-.26355-01
37		-.14033-02
38	XXXXXX	.97135-01
39	XXXX	-.68990-01
40	XXX	-.46423-01
41	XXXX	.40695-01
42		-.63237-03
43	XXXXXXXXX	-.15121-00
44	XXXX	.50106-01
45	XXX	-.38445-01
46	XX	.15374-01
47	XXX	-.33735-01
48		.25520-02
49		.13751-02
50	XXXXX	.80433-01

Figure 2a. Autocorrelation function of residuals from the (0, 3, 3) model

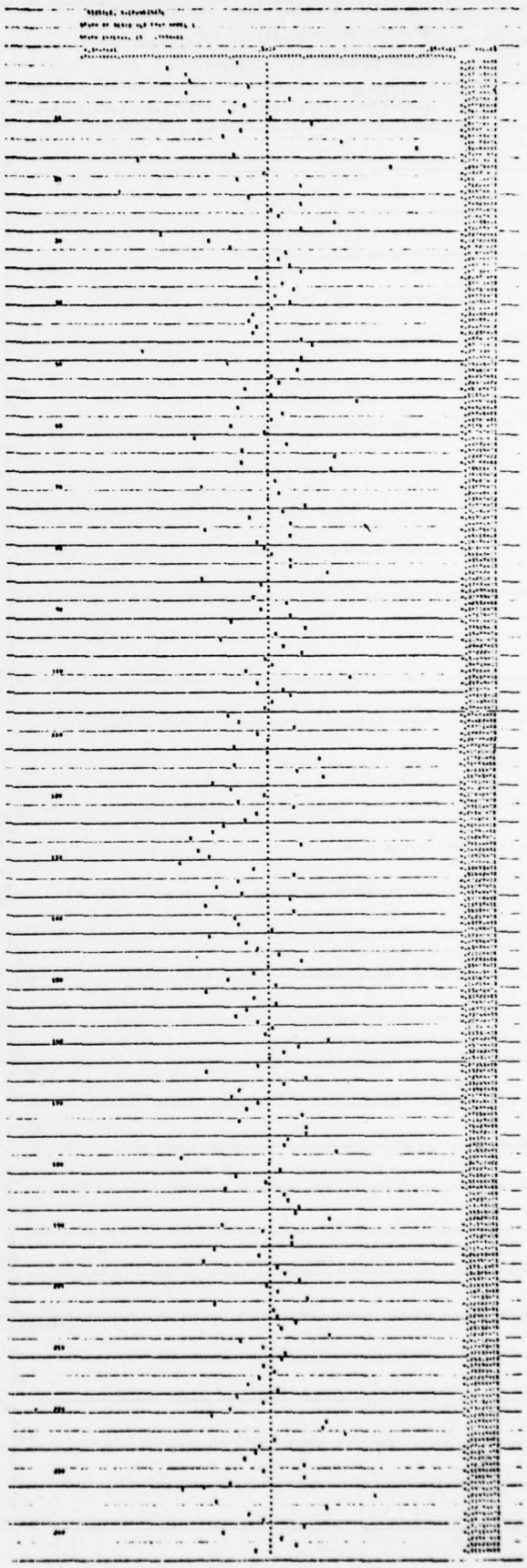


Figure 2b. Residuals from the (0, 3, 5) model

THE ESTIMATED RESIDUALS - MODEL (0,3,5)

GRAPH OF OBSERVED SERIES ACF

GRAPH INTERVAL IS .2000-01

	-.1000+01 .0000 .1000+01	VALUES
1	XXXX	.69365-01
2	X	-.56925-02
3	XXX	.57105-01
4	XX	-.27972-01
5	XX	.18125-01
6	X	.10858-02
7	XX	.24737-01
8	XXX	-.35807-01
9	XX	.16392-01
10	XXX	-.40413-01
11	X	-.65854-02
12	XXXX	.53219-01
13	XXXX	.53401-01
14	XXX	.39992-01
15	X	.96722-03
16	XX	-.25265-01
17	XXXX	.01786-01
18	XX	-.10913-01
19	XXXX	.51654-01
20	XXXX	.52962-01
21	XX	-.25437-01
22	XXX	-.45134-01
23	XXX	.48754-01
24	XXXX	-.50309-01
25	XXX	.39594-01
26	XXX	.44974-01
27	XX	.25559-01
28	XXX	.31274-01
29	XXXX	-.55910-01
30	XXX	-.43708-01
31	XXX	.34629-01
32	XX	.10067-01
33	XXX	-.43998-01
34	XXXXX	-.11902-00
35	XX	-.18481-01
36	XXXX	-.56073-01
37	XX	-.11491-01
38	XXXXX	.70912-01
39	XXX	-.48465-01
40	X	-.40153-02
41	XXXX	.51050-01
42	XX	.12301-01
43	XXXXXXX	-.13521-00
44	XXX	.30973-01
45	XX	-.24522-01
46	XXX	.35957-01
47	XXX	-.46666-01
48	X	.05557-02
49	X	-.82604-02
50	XXX	.44506-01

Figure 2c. Autocorrelation function of residuals from the (0, 3, 5) model

for a sensible model is that the zeroes of this polynomial be outside the unit circle (invertibility property).

It is important to check this and the moduli of the roots given in Table I indicate that the model is indeed invertible.

5. Forecasts

Accepting that the (0, 3, 5) model provides an adequate representation of the system (with the (0, 3, 4) model as a close runner-up) the forecasts produced are most easily calculated from the difference equation itself (see Chapter 5 of B&J). From Equation (5) we find

$$z_t = 3z_{t-1} - 3z_{t-2} + z_{t-3} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \theta_3 a_{t-3} - \theta_4 a_{t-4} - \theta_5 a_{t-5} . \qquad (7)$$

Then by taking conditional expectations of $z_{t+1}, z_{t+2}, \ldots, z_{t+\ell}$ at origin t (as described in B&J p. 130) the $1, 2, 3, \ldots, \ell, \ldots$ step ahead forecasts are:

$$\begin{cases} \hat{z}_t(1) = 3z_t - 3z_{t-1} + z_{t-2} - \theta_1 a_t - \theta_2 a_{t-1} - \theta_3 a_{t-2} - \theta_4 a_{t-3} - \theta_5 a_{t-4} \\ \hat{z}_t(2) = 3\hat{z}_t(1) - 3z_t + z_{t-1} - \theta_2 a_t - \theta_3 a_{t-1} - \theta_4 a_{t-2} - \theta_5 a_{t-3} \\ \hat{z}_t(3) = 3\hat{z}_t(2) - 3\hat{z}_t(1) + z_t - \theta_3 a_t - \theta_4 a_{t-1} - \theta_5 a_{t-2} \\ \hat{z}_t(4) = 3\hat{z}_t(3) - 3\hat{z}_t(2) + \hat{z}(1) - \theta_4 a_t - \theta_5 a_{t-1} \\ \hat{z}_t(5) = 3\hat{z}_t(4) - 3\hat{z}_t(3) - \hat{z}(2) - \theta_5 a_t \\ \hat{z}_t(\ell) = 3\hat{z}_t(\ell-1) - 3\hat{z}(\ell-2) - \hat{z}(\ell-3) \qquad \ell \geq 6 \end{cases} \qquad (8)$$

In practice of course this is done automatically by the computer.

Table II

Obs #	Actual value	Model (0, 3, 5)	Forecasts Model (0, 3, 4)	Model (0, 3, 3)
201	13225.08	13224.78	13224.80	13224.99
202	13306.74	13305.80	13305.94	13306.46
203	13387.51	13386.70	13386.77	13387.51
204	13468.42	13467.20	13467.23	13468.14
205	13549.74	13547.33	13547.34	13548.34
206	13628.61	13627.10	13627.08	13628.12
207	13708.78	13706.49	13706.46	13707.48
208	13788.67	13785.52	13785.48	13786.40
209	13868.21	13864.18	13864.14	13864.91
210	13947.30	13942.47	13942.44	13942.99

For illustration, the forecasts produced by this model with an origin (for all forecasts) at $t = 200$ is shown in Figure 3. It will be noticed that the forecasts are in very close agreement with the actual values. Even the 10 step ahead forecast is hardly distinguishable from the actually observed value.

Table II lists the actual values and the forecasts numerically. The forecasts produced by the models $(0, 3, 4)$ and $(0, 3, 3)$ are also very good and they are included for comparison.

6. Error of Forecasts

In order to determine the error of the forecasts, it is helpful to write the model (1) in random shock form. Thus formally

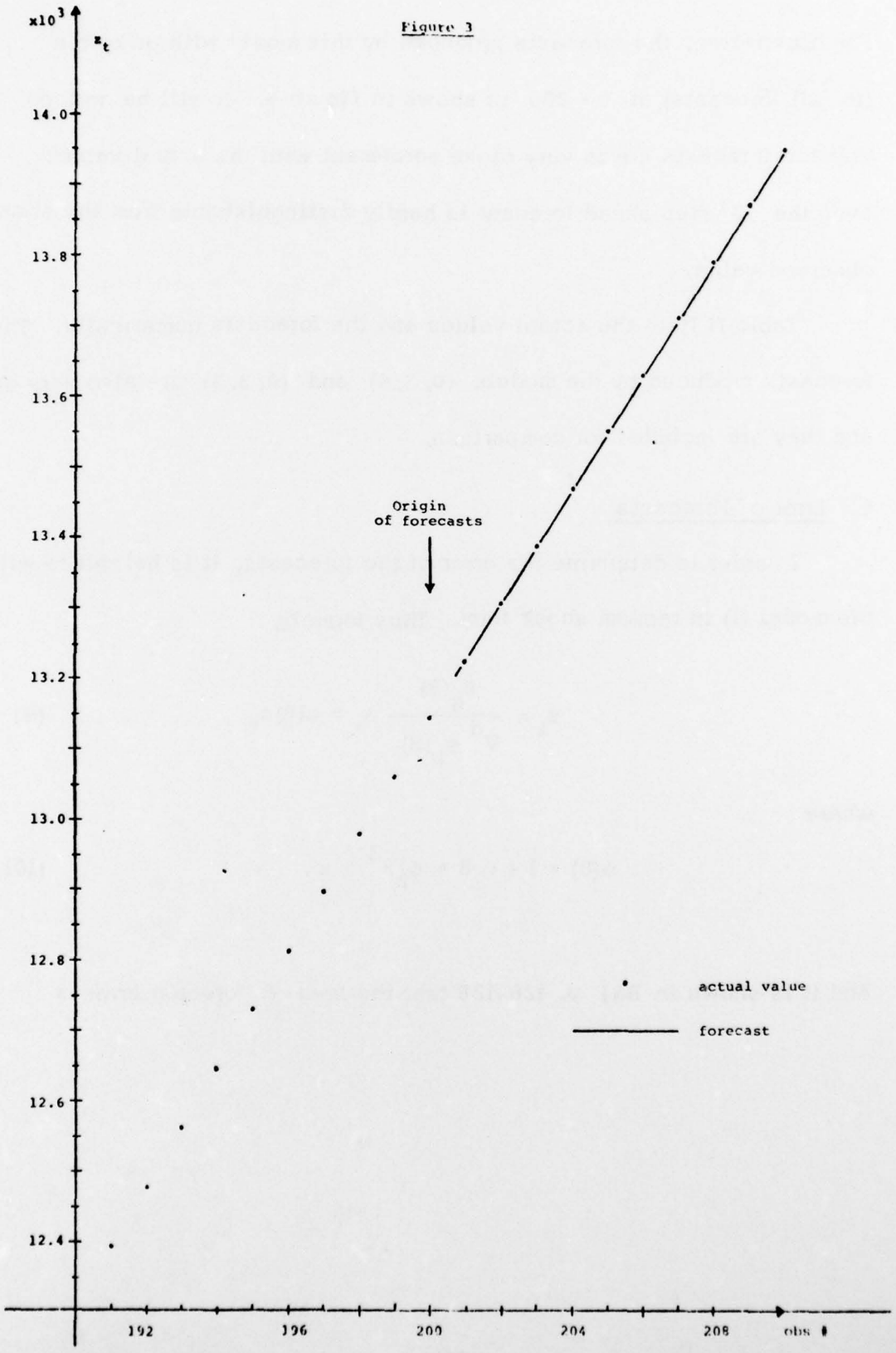
$$z_t = \frac{\theta_q(B)}{\nabla^d \phi_p(B)} a_t = \psi(B) a_t \tag{9}$$

where

$$\psi(B) = 1 + \psi_1 B + \psi_2 B^2 + \ldots \tag{10}$$

And it is shown in B&J p. 126-128 that the lead ℓ forecast error is

Figure 3


x10^3
z_t
14.0
13.8
13.6
13.4
13.2
13.0
12.8
12.6
12.4
Origin
of forecasts
actual value
forecast
192
196
200
204
208
obs #

$$e_t(\ell) = z_{t+\ell} - \hat{z}_t(\ell) = a_{t+\ell} + \psi_1 a_{t+\ell-1} + \ldots + \psi_{\ell-1} a_{t+1} . \quad (13)$$

Whence the variance of the forecast error is

$$\begin{aligned} \text{var}[e_t(\ell)] &= E(z_{t+\ell} - \hat{z}_t(\ell))^2 \\ &= (1 + \psi_1^2 + \psi_2^2 + \ldots + \psi_{\ell-1}^2)\sigma_a^2 . \end{aligned} \quad (14)$$

For the fitted model (5) the ψ-weights are calculated by equating coefficients in (15), B&J pp. 132-134.

$$(1 - 3B + 3B^2 - B^3)(1 + \psi_1 B + \psi_2 B^2 + \ldots) = (1 - \theta_1 B - \theta_2 B^2 - \theta_3 B^3 - \theta_4 B^4 - \theta_5 B^5) \quad (15)$$

Specifically we find the ψ_j values given in Table III. Using the estimated $\hat{\sigma}_a^2 = .81$ from Table I, the variance of the forecast error is given for $\ell = 1, 2, \ldots, 10$. The last column in Table III lists ± 2 standard errors, corresponding to approximately 95% probability intervals for the forecasts. We note, that these probability intervals are so narrow, that they cannot be distinguished from the forecasts themselves in a plot like Figure 3.

All that is needed to compute forecasts and the standard deviations of forecast errors, is given here. What appears in the following Appendices is not necessary for calculation, but does illuminate the nature of the projection process.

Table III

ψ-weights and forecast errors

j	ψ_j	ℓ	$\mathrm{Var}[e_t(\ell)]$	Approx. 95% Probability Intervals
		1	.81	± 1.8
1	.922	2	1.38	± 2.3
2	1.057	3	1.81	± 2.7
3	1.520	4	3.74	± 3.9
4	1.916	5	5.95	± 4.9
5	2.376	6	9.15	± 6.0
6	2.900	7	13.62	± 7.4
7	3.488	8	19.70	± 8.9
8	4.140	9	27.77	± 10.5
9	4.856	10	38.20	± 12.4

Appendix A - Integral forms

As discussed in Chapter 4 pp. 103-114 of B&J, the equivalent integrated form of the model of Equation (5), is of some interest also. In this form the observations appear as a linear aggregates of past random shocks, their difference, sum, sum of sums, etc., plus a new random shock. Specifically the integrated model form

$$z_t = \lambda_{-2} \nabla a_{t-1} + \lambda_{-1} a_{t-1} + \lambda_0 S a_{t-1} + \lambda_1 S^2 a_{t-1} + \lambda_2 S^3 a_{t-1} + a_t \qquad (A\text{-}1)$$

degenerates to different models from Table I when certain of the λ-coefficients are taken to be zero. Table IV links models from Table I to their equivalent integrated forms, and lists estimated λ coefficients which can be calculated from the estimated θ's . Conversion formulas for the models under consideration are given in Table V, but can more generally be found from equating coefficients in Equation 4.3.21, p. 112 in B&J.

Table IV

Integrated model forms

$$z_t = \lambda_{-2} \nabla a_{t-1} + \lambda_{-1} a_{t-1} + \lambda_0 S a_{t-1} + \lambda_1 S^2 a_{t-1} + \lambda_2 S^3 a_{t-1} + a_t$$

Model	λ_{-2}	λ_{-1}	λ_0	λ_1	λ_2	RSS
(0, 2, 2)	-	-	.483	.800	-	410.
(0, 2, 3)	-	.425	.035	.878	-	348.
(0, 3, 3)	-	-	1.104	.016	.149	247.
(0, 3, 4)	-	.173	.627	.197	.065	203.
(0, 3, 5)	.131	-.129	.716	.140	.064	192.

Table V

Conversion formulae θ to λ

Model	Formulae
(0, 2, 2)	$\lambda_0 = 1 + \theta_2$ $\lambda_1 = 1 - \theta_1 - \theta_2$
(0, 2, 3)	$\lambda_{-1} = -\theta_3$ $\lambda_0 = 1 + \theta_2 + 2\theta_3$ $\lambda_1 = 1 - \theta_1 - \theta_2 - \theta_3$
(0, 3, 3)	$\lambda_0 = 1 - \theta_3$ $\lambda_1 = 1 + \theta_2 + 2\theta_3$ $\lambda_2 = 1 - \theta_1 - \theta_2 - \theta_3$
(0, 3, 4)	$\lambda_{-1} = \theta_4$ $\lambda_0 = 1 - \theta_3 - 3\theta_4$ $\lambda_1 = 1 + \theta_2 + 2\theta_3 + 3\theta_4$ $\lambda_2 = 1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$
(0, 3, 5)	$\lambda_{-2} = -\theta_5$ $\lambda_{-1} = \theta_4 + 4\theta_5$ $\lambda_0 = 1 - \theta_3 - 3\theta_4 - 6\theta_5$ $\lambda_1 = 1 + \theta_2 + 2\theta_3 + 3\theta_4 + 4\theta_5$ $\lambda_2 = 1 - \theta_1 - \theta_2 - \theta_3 - \theta_5 - \theta_5$

Appendix B - The eventual forecast function

One question of interest is what function is being selected for projecting the forecasts, i.e. what is the forecast function. It is shown in B&J p. 139 that depending on the nature of the left hand operator, the model (1) could call for forecasts lying on an updating function that could consist of any combination of polynomials, exponentials and sine and cosine waves. What forecast function does the model imply for the present fitted (0, 3, 5) model?

The eventual forecast function for the (0, 3, 5) model satisfies the difference equation

$$\nabla^3 z_t(\ell) = 0 \qquad \text{(B-1)}$$

which has as its solutions a polynomial in ℓ of 2^{nd} degree

$$\hat{z}_t(\ell) = b_0^{(t)} + b_1^{(t)}\ell + b_2^{(t)}\ell^2 \qquad \text{(B-2)}$$

and applies for $\ell > q - p - d$ (i.e. $\ell > 2$) .

In other words the model (0, 3, 5) implies, that the forecasted future values from any time origin t will, except for slight deviations at the first two lead-times, follow a quadratic curve. (The (0, 3, 4) model which fits slightly less well implies that only one initial deviation occurs, while the (0, 3, 3) model implies that all forecasts lie on a quadratic curve).

Although the forecasts are best calculated directly from the difference equation as above it is enlightening to further consider their nature.

As the origin of forecasts is advanced the calculating process requires that coefficients b_0, b_1 and b_2 are sequentially updated. For example the updating formulae for the (0, 3, 5) model can be found directly by relating (B-2) to the forecasting formula from the integrated model.

We find that the updating formulae derived below are

$$\begin{cases} b_0^{(t)} = b_0^{(t-1)} + b_1^{(t-1)} + b_2^{(t-1)} + \lambda_0 a_t \\ b_1^{(t)} = b_1^{(t-1)} + 2b_2^{(t-1)} + (\lambda_1 + \tfrac{1}{2}\lambda_2) a_t \\ b_2^{(t)} = b_2^{(t-1)} + \tfrac{1}{2}\lambda_2 a_t \end{cases} \tag{B-3}$$

Note that the first terms on the right of (B-3) simply allow for movement of the origin without changing the polynomial. The term involving the last random shock a_t appropriately updates the coefficient.

The updating formulae (B-3) are derived as follows. We have from Equation (A-1) in Appendix A that

$$z_{t+\ell} = \lambda_{-2} \nabla a_{t+\ell-1} + \lambda_{-1} a_{t+\ell-1} + \lambda_0 S a_{t+\ell-1} + \lambda_1 S^2 a_{t+\ell-1} + \lambda_2 S^3 a_{t+\ell-1} + a_{t+\ell} \tag{B-4}$$

Assuming $\ell > 2$ and taking expectations at origin t we find

$$\hat{z}_t(\ell) = E(\lambda_0 S a_{t+\ell-1}) + E(\lambda_1 S^2 a_{t+\ell-1}) + E(\lambda_2 S^3 a_{t+\ell-1})$$

$$= (\lambda_0 S a_t) + (\lambda_1 S^2 a_{t-1} + \ell \lambda_1 S a_t)$$

$$+ (\lambda_2 S^3 a_{t-2} + (\ell+1) \lambda_2 S^2 a_{t-1} + \frac{(\ell+1)\ell}{2} \lambda_2 S a_t)$$

$$= (\lambda_0 S a_t + \lambda_1 S^2 a_{t-1} + \lambda_2 S^2 a_{t-1} + \lambda_2 S^3 a_{t-2})$$

$$+ \ell(\lambda_1 S a_t + \lambda_2 S^2 a_{t-1} + \tfrac{1}{2} \lambda_2 S a_t)$$

$$+ \ell^2(\tfrac{1}{2} \lambda_2 S a_t) \qquad \text{(B-5)}$$

The coefficients b_0, b_1, b_2 in Equation (B-2) are now identified as

$$\begin{cases} b_0^{(t)} = \lambda_0 S a_t + \lambda_1 S^2 a_{t-1} + \lambda_2 S^2 a_{t-1} + \lambda_2 S^3 a_{t-2} \\ b_1^{(t)} = \lambda_1 S a_t + \lambda_2 S^2 a_{t-1} + \tfrac{1}{2} \lambda_2 S a_t \\ b_2^{(t)} = \tfrac{1}{2} \lambda_2 S a_t \end{cases} \qquad \text{(B-6)}$$

Now it is seen that (B-6) can be rewritten as (B-3).

Appendix C - How are the data used in the forecast?

Still another way to interpret the forecasts is as a weighted sum of previous observations: Writing (5) as

$$\frac{\nabla^3}{\theta(B)}\, z_t = \pi(B)\, z_t = a_t \qquad \text{(C-1)}$$

where

$$\pi(B) = 1 - \pi_1 B - \pi_2 B^2 - \ldots \qquad \text{(C-2)}$$

we find that

$$\hat{z}_t(\ell) = \pi_1 \hat{z}_t(\ell-1) + \pi_2 \hat{z}_t(\ell-2) + \ldots \qquad \text{(C-3)}$$

where $\hat{z}_t(-h)$ is taken to mean z_{t-h} for $h = 0, 1, 2, \ldots$. The π-weights can be found by equating coefficients in the following identity after the θ-estimates have been substituted:

$$\nabla^3 = \theta(B)\,\pi(B)$$

$$(1 - 3B + 3B^2 - B^3) = (1 - \theta_1 B - \theta_2 B^2 - \theta_3 B^3 - \theta_4 B^4 - \theta_5 B^5)(1 - \pi_1 B - \pi_2 B^2 - \pi_3 B^3 - \ldots) \qquad \text{(C-4)}$$

The π-weights (also denoted by $\pi^{(1)}$) are given in Figure 4; thus for example

$$\begin{aligned}\hat{z}_t(1) = {} & .922 \times z_t + .207 \times z_{t-1} + .355 \times z_{t-2} - .039 \times z_{t-3} \\ & - .068 \times z_{t-4} - .171 \times z_{t-5} - .149 \times z_{t-6} - .143 \times z_{t-7} \\ & - .107 \times z_{t-8} - .079 \times z_{t-9} - .046 \times z_{t-10} - .018 \times z_{t-11} \\ & + .008 \times z_{t-12} + .027 \times z_{t-13} + .041 \times z_{t-14} + .048 \times z_{t-15} \\ & + .048 \times z_{t-16} + .044 \times z_{t-17} + .036 \times z_{t-18} + .026 \times z_{t-20} + \ldots\end{aligned} \qquad \text{(C-5)}$$

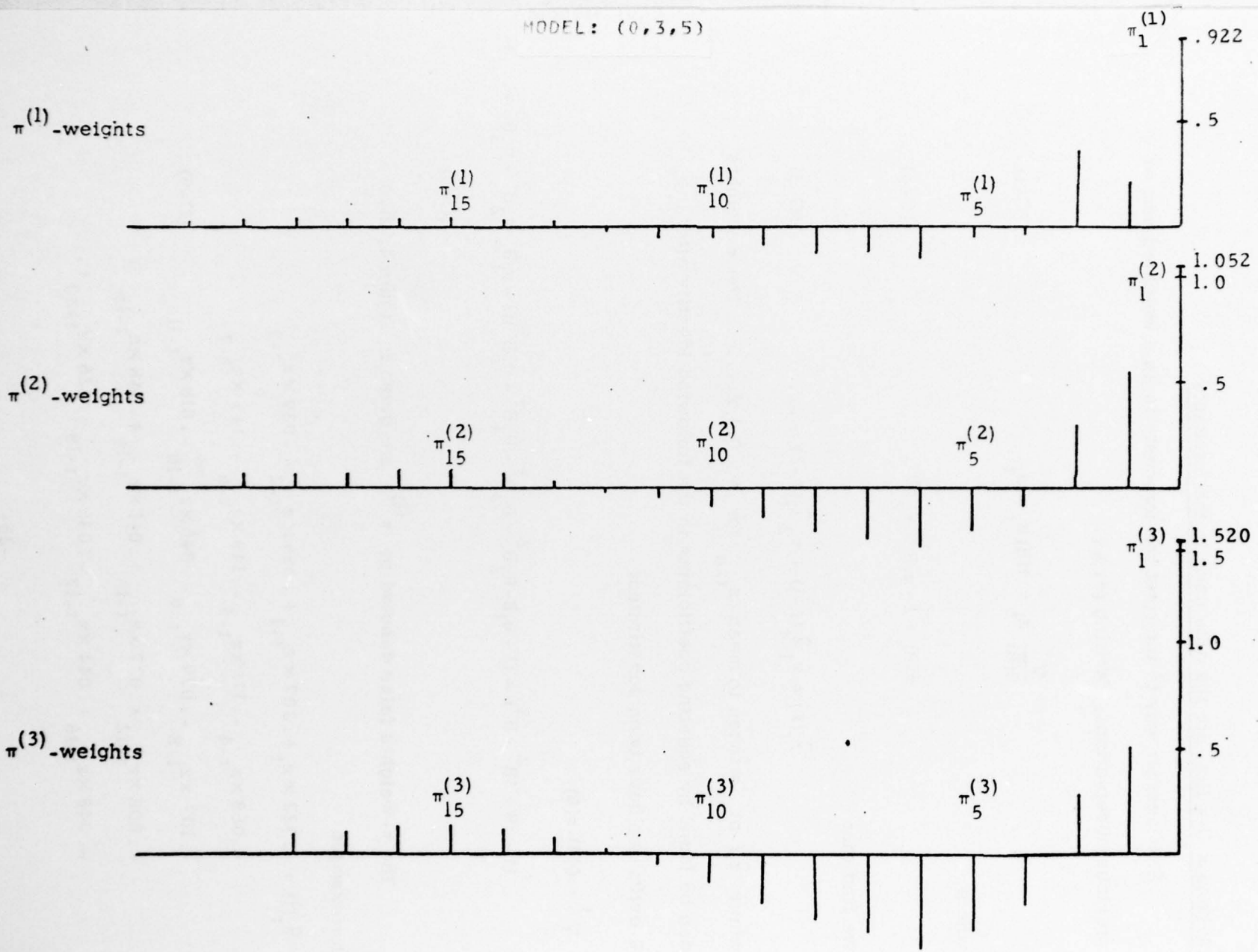


Figure 4. π-weights for the (0, 3, 5) model

The two step ahead forecast can be found similarly by replacing z_t by $\hat{z}_t(1)$ and z_{t-j} by z_{t-j+1}, and so on for forecasts with higher lead times. However these forecasts may also be expressed directly as weighted sums of the observations $z_t, z_{t-1}, z_{t-2}, \ldots$. The weights $\pi^{(2)}$ and $\pi^{(3)}$ corresponding to the two and three step ahead forecasts respectively, are also show in Figure 4. In general these weights may be found from the $\pi^{(1)}$-weight by means of the formula (p. 142 Equation 5.3.9)

$$\pi_j^{(\ell)} = \pi_{j+\ell-1} + \sum_{h=1}^{\ell-1} \pi_h \pi_j^{(\ell-h)} . \qquad \text{(C-6)}$$

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Data projection (forecasting) methods developed in the book "Time Series Analysis Forecasting and Control" by Box and Jenkins (published by Holden-Day revised edition 1976) are illustrated using missile data supplied by Quality Evaluation Division of White Sands Missile Range. The process of model building making iterative use of identification (using the autocorrelation function) → next page

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fitting (maximum likelihood estimation) and diagnostic checking (analysis residuals) is illustrated in the building of an appropriate stochastic difference equation model.

It is shown in detail how all the following may be calculated directly from the model:

1) the projection (forecast) function,
2) the memory function,
3) the error of the projected value at any lead time, and
4) the updating of the projection function.